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The Girardeau's fermion-boson procedure in the light of the composite-boson many-body theory

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Abstract. We reconsider the procedure developed for atoms a few decades ago by Girardeau, in the light of the composite-boson many-body theory we recently proposed. The Girardeau's procedure makes use of a so called "unitary Fock-Tani operator" which in an exact way transforms *one* composite bound atom into one bosonic "ideal" atom. When used to transform the Hamiltonian of interacting atoms, this operator generates an extremely complex set of effective scatterings between ideal bosonic atoms and free fermions which makes the transformed Hamiltonian impossible to write explicitly, in this way forcing to some truncation. The scatterings restricted to the ideal-atom subspace are shown to read rather simply in terms of the two elementary scatterings of the composite-boson many-body theory, namely, the energy-like direct interaction scatterings — which describe fermion interactions without fermion exchange — and the dimensionless Pauli scatterings — which describe fermion exchanges without fermion interaction. We here show that, due to a fundamental difference in the scalar products of elementary and composite bosons, the Hamiltonian expectation value for N ground state atoms obtained by staying in the ideal-atom subspace and working with boson operators only, differ from the exact ones even for $N = 2$ and a mapping to the ideal-atom subspace performed, as advocated, from the fully antisymmetrical atomic state, i.e., the state which obeys the so-called "subsidiary condition". This shows that, within this Girardeau's procedure too, we cannot completely forget the underlying fermionic components of the particles if we want to correctly describe their interactions.

PACS. 05.30.Jp Boson systems – 34.50.-s Scattering of atoms and molecules – 71.35.-y Excitons and related phenomena

1 Introduction

Although atoms are known to be made of many fermions, they are commonly treated as elementary quantum particles. The deep motivation for such a simplification can be seen as barely practical: as the many-body procedures up to now available are valid for elementary fermions or elementary bosons only [1,2], it was somehow a necessity to replace composite atoms by elementary quantum particles, in order to deal with their interactions.

Such a replacement corresponds to freeze the fermionic components of these composite atoms in a given configuration once and for all, in order to, afterwards, forget all fermion exchanges between atoms. The physical argument usually given to support this exchange omission is the fact that, in the low density limit, the atoms are very far apart so that the overlap between atomic wave functions is very small, making the exchanges between atoms also very small. This argument is not correct for atoms cold enough to behave as quantum particles: indeed, the center of mass of these composite atoms made of elementary fermions is not pinned in a well defined position, but delocalized over the whole sample — or, more realistically, over the atom coherence length — so that atoms are never "far apart". In saying that atoms are far apart, we see them not as quantum particles but as wave packets, the energy dispersion of these wave packets being small compared to k_BT — which is what happens in most experimental conditions, for the mass of the atom center-of-mass is quite large. However, such wave packets are not appropriate to describe interacting atoms when the temperature is very low, as in the experiments on ultracold gas nowadays performed.

Various mathematical procedures [3] have been proposed to map the two-fermion space into the elementary-boson space, i.e., to replace atoms made of two fermions by elementary bosons. These bosonization procedures include a certain amount of fermion exchanges in the effective scatterings between "ideal" elementary atoms they generate. Girardeau has tackled this problem a few decades ago [4–12] through a procedure conceptually more

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sophisticated than most other approaches [13], as it formally contains the possible dissociations of bosonic "ideal" atoms into their fermionic components. The method, which is still considered as a relevant approach to composite bosons, is based on a smart unitary transformation — called the Fock-Tani transformation by Girardeau [14] — which maps the bound states of a composite atom into a different space made of "ideal elementary atoms" in a fully *exact* manner in the case of *just one* composite atom. This unitary transformation is then used to formally transform the Hamiltonian of interacting atoms, originally written in terms of fermion operators, into a Hamiltonian which, in addition to fermion operators, also contains elementary-boson operators which represent the "ideal atoms". The two-body scatterings between these ideal atoms are found to contain direct as well as exchange Coulomb processes. This procedure also generates very many scatterings between ideal atoms and fermions, which are increasingly complex. As a direct consequence, the transformed Hamiltonian cannot be written explicitly, thus forcing to some drastic truncation.

A few years ago, the author, along with Betbeder-Matibet, has tackled the problem of interacting composite excitons through a completely different approach [15]. (Excitons differ from hydrogen atoms for their mass ratio only, if we forget "electron-hole exchange", i.e., Coulomb scatterings between the valence and conduction bands.) The challenge was to completely avoid the introduction of elementary-boson operators while proposing an exact but tractable way to deal with fermion exchanges between composite quantum particles. This composite-boson many-body theory [15], recalled in Section 2, makes use of the linear combinations of one-fermion-pair states which are the exact *bound and unbound* eigenstates of the system Hamiltonian. It relies on four nicely simple commutators between the operators creating these fermion pairs: two for Coulomb interactions without exchange process and two for fermion exchanges without interaction process. The so-called Shiva diagrams [16] associated to this theory allow us to visualize the tricky fermion exchanges which take place between these composite quantum particles. Using them, it is now possible not only to see but also to readily calculate any physical effect involving interacting excitons, — or atoms made of fermion pairs — with the fermion exchanges between any number of these composite particles included in an exact way. In particular, these Shiva diagrams make easy the selection of all contributions leading to a particular order in the composite-boson density,

$$\eta = n\, a_B^D = N \left(\frac{a_B}{L}\right)^D , \tag{1}$$

where N is the composite-boson number, a_B the spatial extension of the composite-boson bound state of interest, L the sample size and D the space dimension. η is the small dimensionless parameter that emerges in all low-density properties of these many-body systems. Let us stress that these η expansions are rather tricky to perform, a priori, because they are *not*, as usual, driven by the number of interactions since these fermion interactions are often responsible for the composite-boson bound states.

In view of the popularity that the Girardeau's approach has gained over the last decades in atomic [17–20] and also in nuclear physics [21–23], it is of interest to revisit this sophisticated procedure in the light of the new composite-boson many-body theory. In order to make clear the basic ideas of Girardeau's approach, we have decided to stick on the original work described in reference [7], although some important improvements have been later on proposed through a generalized Fock-Tani transformation [11]. They however lead to calculations which are even more complicated. This original work is briefly recalled in Section 3. A precise rewriting of the part of the transformed Hamiltonian between ideal atoms is made in terms of the scatterings appearing in the composite-boson many-body theory described in Section 2. Section 4 is devoted to a discussion of some fundamental differences which exist between the Girardeau's approach of reference [7] and the composite-boson many-body theory. This section also points out some important consequences of these differences in a physically relevant quantity, namely, the Hamiltonian expectation value for N ground state atoms at low density, these fundamental differences remaining in the $N = 2$ limit.

As a main result, we here show that, although the unitary Fock-Tani transformation presented in reference [7] should in principle allow us to recover the exact results derived by the composite-boson many-body theory *if all the terms are kept* — since the results cannot be changed by introducing the unity operator I written as $U^{-1}U$ — its restriction to the bosonic-ideal-atom subspace leads to incorrect results for $N = 2$ interacting atoms already, *even* if the boson mapping is done, as advocated, from the fully antisymmetrical state, i.e., the state which obeys the so-called "subsidiary condition". This strongly reduces the interest of the Girardeau's procedure because, when all the terms of the transformed Hamiltonian *and* all the terms of the transformed states are kept, this procedure, even in its simplest form, i.e., without using the generalized Fock-Tani transformation proposed later on [11], is far less compact than the composite-boson many-body theory we have constructed.

In the present paper, we consider a system made of two different species of fermionic particles, in equal number, with an attractive potential between them strong enough to have bound states. In the following, we are going to call *coboson* — a contraction for "composite boson" — a pair of interacting fermions, *this pair being either in a bound or in an extended state*. In order to make an easy link between the present paper and the various previous works, we have chosen to keep the notations of these previous works, while giving the precise correspondence which exists between them. These two sets of notations may however lead to some confusion: as an example, the two fermion species are called α and β in the composite-boson many-body theory, while α labels the bound atom states in the Girardeau's procedure described in reference [7].

2 Survey of the composite-boson many-body theory

2.1 The system Hamiltonian

The Hamiltonian of a translationally invariant system made of fermions α and β reads in first quantization as

$$H = \sum_i \frac{\mathbf{p}_{\alpha_i}^2}{2m_\alpha} + \sum_j \frac{\mathbf{p}_{\beta_j}^2}{2m_\beta} + \frac{1}{2}\sum_{i\neq i'} v_{\alpha\alpha}(\mathbf{r}_{\alpha_i} - \mathbf{r}_{\alpha_{i'}}) + \frac{1}{2}\sum_{j\neq j'} v_{\beta\beta}(\mathbf{r}_{\beta_j} - \mathbf{r}_{\beta_{j'}}) + \sum_{i,j} v_{\alpha\beta}(\mathbf{r}_{\alpha_i} - \mathbf{r}_{\beta_j}). \quad (2)$$

In the case of hydrogen atoms, the fermions α are the electrons and the fermions β the protons, so that $m_\alpha \ll m_\beta$. In the case of excitons, the fermions α are the conduction electrons, i.e., electrons dressed by their interactions with the periodic semiconductor crystal, while the fermions β are the valence holes, so that m_α is of the order of m_β. The fermions of the H atoms or the fermions of the excitons interact through the Coulomb potential so that $v_{\alpha\alpha}(\mathbf{r}) = v_{\beta\beta}(\mathbf{r}) = -v_{\alpha\beta}(\mathbf{r}) = e^2/\varepsilon_r r$, where ε_r is the semiconductor relative dielectric constant in the case of excitons while $\varepsilon_r = 1$ for H atoms in vacuum.

It is convenient to write this Hamiltonian in second quantization using the free-fermion states $|\mathbf{k}_\alpha\rangle = a^\dagger_{\mathbf{k}_\alpha}|0\rangle$ and $|\mathbf{k}_\beta\rangle = a^\dagger_{\mathbf{k}_\beta}|0\rangle$ defined as

$$\left(\frac{\mathbf{p}_\alpha^2}{2m_\alpha} - \epsilon^{(\alpha)}_{\mathbf{k}_\alpha}\right)|\mathbf{k}_\alpha\rangle = 0$$
$$\left(\frac{\mathbf{p}_\beta^2}{2m_\beta} - \epsilon^{(\beta)}_{\mathbf{k}_\beta}\right)|\mathbf{k}_\beta\rangle = 0, \quad (3)$$

with $\epsilon^{(\alpha)}_{\mathbf{k}_\alpha} = k_\alpha^2/2m_\alpha$ and $\epsilon^{(\beta)}_{\mathbf{k}_\beta} = k_\beta^2/2m_\beta$.

Since holes are valence-electron absences, their creation and destruction operators $b^\dagger_{\mathbf{k}_\beta}$ and $b_{\mathbf{k}_\beta}$ unambiguously anticommute with the creation operators of conduction electrons $a^\dagger_{\mathbf{k}}$. It is convenient to consider that the operator $a^\dagger_{\mathbf{k}_\alpha}$ also anticommutes with $b_{\mathbf{k}_\beta}$ or $b^\dagger_{\mathbf{k}_\beta}$ in the case of electrons and protons. This allows us to avoid irrelevant minus signs which appear all over the calculation but disappear from all physical results — as possible to explicitly check. Consequently, the anticommutation relations for the fermion operators $a^\dagger_{\mathbf{k}_\alpha}$ and $b^\dagger_{\mathbf{k}_\beta}$ read, for H atoms as well as for excitons, as

$$\left[a_{\mathbf{k}_\alpha}, a^\dagger_{\mathbf{k}'_\alpha}\right]_+ = \delta_{\mathbf{k}_\alpha,\mathbf{k}'_\alpha}\ , \qquad \left[a_{\mathbf{k}_\alpha}, a_{\mathbf{k}'_\alpha}\right]_+ = 0\ ,$$
$$\left[b_{\mathbf{k}_\beta}, b^\dagger_{\mathbf{k}'_\beta}\right]_+ = \delta_{\mathbf{k}_\beta,\mathbf{k}'_\beta}\ , \qquad \left[b_{\mathbf{k}_\beta}, b_{\mathbf{k}'_\beta}\right]_+ = 0\ ,$$
$$\left[a_{\mathbf{k}_\alpha}, b^\dagger_{\mathbf{k}'_\beta}\right]_+ = \left[a_{\mathbf{k}_\alpha}, b_{\mathbf{k}'_\beta}\right]_+ = 0, \quad (4)$$

the momenta $\mathbf{k}_\alpha$ and $\mathbf{k}_\beta$ being quantized in units $2\pi/L$ for fermions in a finite volume L^3, in order for the fermion wave functions to be normalized.

The system Hamiltonian in second quantization then reads

$$H = H_\alpha + H_\beta + V_{\alpha\alpha} + V_{\beta\beta} + V_{\alpha\beta}, \quad (5)$$

with the kinetic parts given by

$$H_\alpha = \sum_{\mathbf{k}_\alpha} \epsilon^{(\alpha)}_{\mathbf{k}_\alpha} a^\dagger_{\mathbf{k}_\alpha} a_{\mathbf{k}_\alpha}\ , \qquad H_\beta = \sum_{\mathbf{k}_\beta} \epsilon^{(\beta)}_{\mathbf{k}_\beta} b^\dagger_{\mathbf{k}_\beta} b_{\mathbf{k}_\beta}, \quad (6)$$

while the Coulomb parts are given by

$$V_{\alpha\beta} = -\sum_{\mathbf{q}\neq 0} V_{\mathbf{q}} \sum_{\mathbf{k}_\alpha,\mathbf{k}_\beta} a^\dagger_{\mathbf{k}_\alpha+\mathbf{q}} b^\dagger_{\mathbf{k}_\beta-\mathbf{q}} b_{\mathbf{k}_\beta} a_{\mathbf{k}_\alpha}, \quad (7)$$

$$V_{\alpha\alpha} = \frac{1}{2}\sum_{\mathbf{q}\neq 0} V_{\mathbf{q}} \sum_{\mathbf{k}_{\alpha_1},\mathbf{k}_{\alpha_2}} a^\dagger_{\mathbf{k}_{\alpha_1}+\mathbf{q}} a^\dagger_{\mathbf{k}_{\alpha_2}-\mathbf{q}} a_{\mathbf{k}_{\alpha_2}} a_{\mathbf{k}_{\alpha_1}}, \quad (8)$$

and similarly for $V_{\beta\beta}$, the Fourier transform of the Coulomb potential being $V_{\mathbf{q}} = 4\pi e^2/\varepsilon_r L^3 q^2$ in a finite 3D volume L^3, and $V_{\mathbf{q}} = 2\pi e^2/\varepsilon_r L^2 q^2$ in a quantum well with 2D area L^2.

2.2 The coboson operators $\mathbf{B}^\dagger_\mathbf{i}$

Let us introduce the creation operators for the one-fermion-pair eigenstates of the system Hamiltonian,

$$H|i\rangle \equiv H\, B^\dagger_i|v\rangle = E_i|i\rangle, \quad (9)$$

the set of $|i\rangle$'s spanning *all* bound and unbound one-pair states.

Since the $|i\rangle$'s are the one-pair eigenstates of H, while the $|\mathbf{k}_\alpha,\mathbf{k}_\beta\rangle$'s are the one-pair eigenstates of $H_\alpha+H_\beta$, they both form a complete basis for one-pair states, so that

$$I = \sum_i |i\rangle\langle i|,$$
$$I = \sum_{\mathbf{k}_\alpha,\mathbf{k}_\beta} |\mathbf{k}_\alpha,\mathbf{k}_\beta\rangle\langle\mathbf{k}_\beta,\mathbf{k}_\alpha|. \quad (10)$$

The above equations allow us to expand cobosons on free pairs as

$$|i\rangle = \sum_{\mathbf{k}_\alpha,\mathbf{k}_\beta} |\mathbf{k}_\alpha,\mathbf{k}_\beta\rangle\langle\mathbf{k}_\beta,\mathbf{k}_\alpha|i\rangle,$$

and free pairs on cobosons in a similar way. Using these expansions, we readily find that the creation operators for interacting and free pairs are linked by

$$B^\dagger_i = \sum_{\mathbf{k}_\alpha,\mathbf{k}_\beta} a^\dagger_{\mathbf{k}_\alpha} b^\dagger_{\mathbf{k}_\beta} \langle\mathbf{k}_\beta,\mathbf{k}_\alpha|i\rangle, \quad (11)$$

$$a^\dagger_{\mathbf{k}_\alpha} b^\dagger_{\mathbf{k}_\beta} = \sum_i B^\dagger_i \langle i|\mathbf{k}_\alpha,\mathbf{k}_\beta\rangle. \quad (12)$$

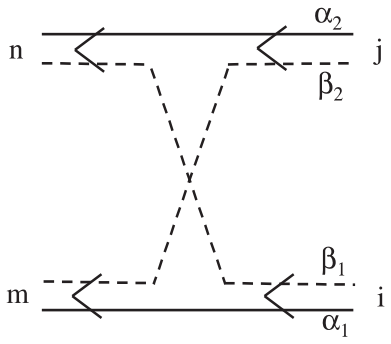

Fig. 1. Pauli scattering $\lambda \binom{n\ j}{m\ i}$ between two cobosons in states i and j, as given in equation (15), i.e., fermion exchange in the absence of fermion interaction. Solid line: fermion α. Dashed line: fermion β.

2.3 Exchanges between cobosons

Using the commutation relations (4), it is easy to show that [24]

$$\left[B_m, B_i^\dagger\right] = \delta_{m,i} - D_{mi}, \tag{13}$$

$$\left[D_{mi}, B_j^\dagger\right] = \sum_n \left\{\lambda \binom{n\ j}{m\ i} + \lambda \binom{m\ j}{n\ i}\right\} B_n^\dagger. \tag{14}$$

where the exchange (or Pauli) scatterings $\lambda \binom{n\ j}{m\ i}$, defined in equation (14) and shown in Figure 1, are dimensionless parameters which correspond to exchange a fermion β between the cobosons i and j, the cobosons m and i of the lower line by definition having the same fermion α. The amplitude of this scattering is given by (see Ref. [25] for a detailed derivation)

$$\lambda \binom{n\ j}{m\ i} = \int d\mathbf{r}_{\alpha_1}\, d\mathbf{r}_{\alpha_2}\, d\mathbf{r}_{\beta_1}\, d\mathbf{r}_{\beta_2}\, \phi_m^*(\mathbf{r}_{\alpha_1}, \mathbf{r}_{\beta_2}) \times \phi_n^*(\mathbf{r}_{\alpha_2}, \mathbf{r}_{\beta_1})\, \phi_i(\mathbf{r}_{\alpha_1}, \mathbf{r}_{\beta_1})\, \phi_j(\mathbf{r}_{\alpha_2}, \mathbf{r}_{\beta_2}). \tag{15}$$

An important property of the "deviation-from-boson operator" is $D_{mi}|v\rangle = 0$, as easy to show by making equation (13) acting on vacuum.

From the commutators (13,14), we readily find [26]

$$\langle 0|B_m B_n B_i^\dagger B_j^\dagger|0\rangle = \delta_{m,i}\,\delta_{n,j} - \lambda \binom{n\ j}{m\ i} + (m \leftrightarrow n). \tag{16}$$

This shows that the N-coboson states are not orthogonal for $N = 2$ already.

Using these commutators, we can also calculate the normalization factor for a N-coboson state having all its cobosons in the same state 0. A less trivial calculation leads to [27,28]

$$\langle 0|B_0^N B_0^{\dagger N}|0\rangle = N! F_N, \tag{17}$$

where F_N obeys the recursion relation [16,28]

$$F_N = F_{N-1} - (N-1) F_{N-2} \lambda_2 \begin{pmatrix} 0\ 0 \\ 0\ 0 \end{pmatrix} + (N-1)(N-2) F_{N-3} \lambda_3 \begin{pmatrix} 0\ 0 \\ 0\ 0 \\ 0\ 0 \end{pmatrix} - \cdots \tag{18}$$

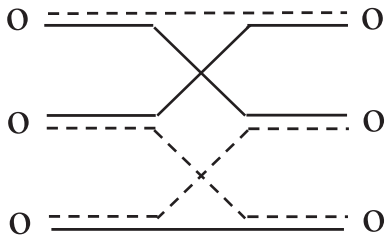

Fig. 2. Shiva diagram for the fermion exchange λ_3 between 3 cobosons in state 0, leading to 3 cobosons in state 0, as appearing in equation (18).

λ_2 is the exchange scattering λ between two cobosons defined in equation (15), while λ_3 corresponds to the exchange betweeen 3 cobosons shown in the Shiva diagram of Figure 2, the higher-order prefactors λ_n being represented by similar diagrams. Their precise values can be found in reference [16].

From equation (18), it is possible to show [27,28] that, in the small-density limit, F_N is *not* close to 1, as could be naïvely thought, but exponentially small. Indeed, for 0 being the hydrogen atom or exciton ground state in 3D, we find, using equation (15),

$$\lambda \begin{pmatrix} 0\ 0 \\ 0\ 0 \end{pmatrix} = \frac{33\pi}{2} \left(\frac{a_B}{L}\right)^3, \tag{19}$$

where $a_B = \hbar^2 \varepsilon_r (m_\alpha^{-1} + m_\beta^{-1})/e^2$ is the coboson Bohr radius. According to equation (18), this leads to a "corrective" factor F_N which reads

$$F_N \simeq \exp\left[N\left(-\frac{33\pi}{4}\eta + O(\eta^2)\right)\right], \tag{20}$$

with η defined in equation (1). While, in the small density limit, $\eta \ll 1$ (as necessary for bound cobosons not to dissociate into a dense two-fermion plasma), the product $N\eta$ is usually large in macroscopic sample, which makes F_N exponentially small.

2.4 Interactions between cobosons

Interactions *between* cobosons are not easy to define properly due to the composite nature of the particles : indeed, $V_{\alpha\beta'}$ is part of the interactions between cobosons if the cobosons are made of the fermions (α, β) and (α', β'), while the same $V_{\alpha\beta'}$ is an interaction inside one of the cobosons if these cobosons are made of (α, β') and $(\alpha'\beta)$. Due to the fermion undistinguishability, there is no way to know how the cobosons are made. Consequently, there is no way to write the interactions betwen composite quantum particles as a potential.

Interactions between cobosons can actually be handled, not through a potential, but through the set of "creation-potentials" $V_i^\dagger$ for the coboson i. These creation-potentials emerge from the commutator [24]

$$\left[H, B_i^\dagger\right] = E_i B_i^\dagger + V_i^\dagger. \tag{21}$$

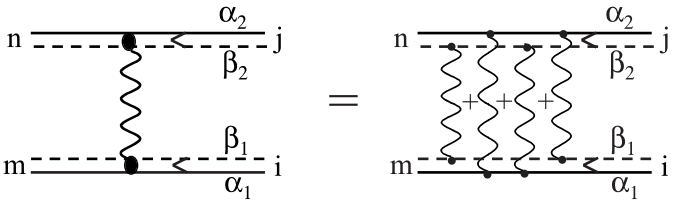

Fig. 3. Direct interaction scattering $\xi^{\rm dir}\left(\begin{smallmatrix} n & j \\ m & i \end{smallmatrix}\right)$, as given in equation (23), due to the fermion interaction between cobosons in states i and j, in the absence of exchange process.

The direct scatterings between two cobosons then follow from

$$\left[V_i^\dagger, B_j^\dagger\right] = \sum_{mn} \xi^{\rm dir}\left(\begin{smallmatrix} n & j \\ m & i \end{smallmatrix}\right) B_m^\dagger B_n^\dagger. \tag{22}$$

These scatterings, shown in Figure 3, are energy-like quantities given by (see Ref. [25] for a detailed derivation)

$$\xi^{\rm dir}\left(\begin{smallmatrix} n & j \\ m & i \end{smallmatrix}\right) = \int d\mathbf{r}_{\alpha_1}\, d\mathbf{r}_{\alpha_2}\, d\mathbf{r}_{\beta_1}\, d\mathbf{r}_{\beta_2}\, \phi_m^*(\mathbf{r}_{\alpha_1},\mathbf{r}_{\beta_1}) \times \phi_n^*(\mathbf{r}_{\alpha_2},\mathbf{r}_{\beta_2})\, \phi_i(\mathbf{r}_{\alpha_1},\mathbf{r}_{\beta_1})\, \phi_j(\mathbf{r}_{\alpha_2},\mathbf{r}_{\beta_2}) \times [v_{\alpha\alpha}(\mathbf{r}_{\alpha_1}-\mathbf{r}_{\alpha_2}) + v_{\beta\beta}(\mathbf{r}_{\beta_1}-\mathbf{r}_{\beta_2}) + v_{\alpha\beta}(\mathbf{r}_{\alpha_1}-\mathbf{r}_{\beta_2}) + v_{\alpha\beta}(\mathbf{r}_{\alpha_2}-\mathbf{r}_{\beta_1})]. \tag{23}$$

Note that, for $v_{\alpha\alpha} = -v_{\alpha\beta}$, as in the case of oppositely charged fermions interacting through Coulomb potentials, we have

$$\xi^{\rm dir}\left(\begin{smallmatrix} n & j \\ i & i \end{smallmatrix}\right) = 0, \tag{24}$$

whatever the parity of the coboson state i is, as seen by exchanging α and β in equation (23).

An important property of the creation-potential is $V_i^\dagger|v\rangle = 0$, as easy to show by making equation (21) acting on vacuum.

Using the above commutators (21, 22) as well as the scalar product (16), it is straightforward to derive the matrix elements of the system Hamiltonian H in the two-coboson subspace. For H acting on the right, we find [24]

$$\langle 0|B_m B_n H B_i^\dagger B_j^\dagger|0\rangle = (E_i+E_j)\left[\delta_{m,i}\,\delta_{n,j} - \lambda\left(\begin{smallmatrix} n & j \\ m & i \end{smallmatrix}\right)\right] + \xi^{\rm dir}\left(\begin{smallmatrix} n & j \\ m & i \end{smallmatrix}\right) - \xi^{\rm in}\left(\begin{smallmatrix} n & j \\ m & i \end{smallmatrix}\right) + (m \leftrightarrow n), \tag{25}$$

where $\xi^{\rm in}$ is the exchange interaction scattering shown in Figure 4, and defined as (see Ref. [25] for a detailed derivation)

$$\xi^{\rm in}\left(\begin{smallmatrix} n & j \\ m & i \end{smallmatrix}\right) = \sum_{pq} \lambda\left(\begin{smallmatrix} n & q \\ m & p \end{smallmatrix}\right) \xi^{\rm dir}\left(\begin{smallmatrix} q & j \\ p & i \end{smallmatrix}\right) \tag{26}$$

$$= \int d\mathbf{r}_{\alpha_1}\, d\mathbf{r}_{\alpha_2}\, d\mathbf{r}_{\beta_1}\, d\mathbf{r}_{\beta_2}\, \phi_m^*(\mathbf{r}_{\alpha_1},\mathbf{r}_{\beta_2}) \times \phi_n^*(\mathbf{r}_{\alpha_2},\mathbf{r}_{\beta_1})\, \phi_i(\mathbf{r}_{\alpha_1},\mathbf{r}_{\beta_1})\, \phi_j(\mathbf{r}_{\alpha_2},\mathbf{r}_{\beta_2}) \times [v_{\alpha\alpha}(\mathbf{r}_{\alpha_1}-\mathbf{r}_{\alpha_2}) + v_{\beta\beta}(\mathbf{r}_{\beta_1}-\mathbf{r}_{\beta_2}) + v_{\alpha\beta}(\mathbf{r}_{\alpha_1}-\mathbf{r}_{\beta_2}) + v_{\alpha\beta}(\mathbf{r}_{\alpha_2}-\mathbf{r}_{\beta_1})], \tag{27}$$

the interactions being between the "in" cobosons (i,j), but inside the "out" cobosons (m,n).

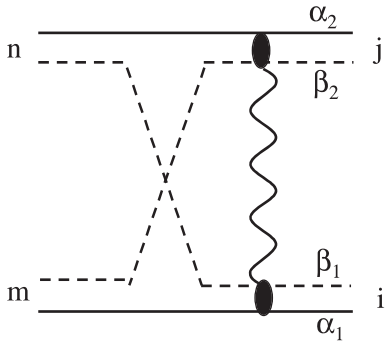

Fig. 4. Exchange interaction scattering $\xi^{\rm in}\left(\begin{smallmatrix} n & j \\ m & i \end{smallmatrix}\right)$, as given in equation (27), due to fermion interactions between cobosons in states i and j followed by a fermion exchange.

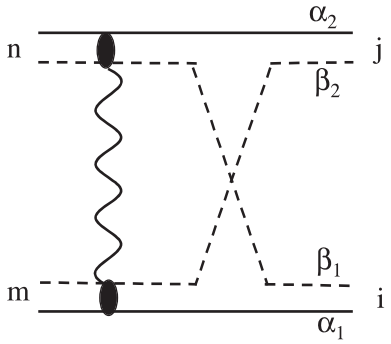

Fig. 5. Exchange interaction scattering $\xi^{\rm out}\left(\begin{smallmatrix} n & j \\ m & i \end{smallmatrix}\right)$, as given in equation (29), due to a fermion exchange between cobosons in states i and j followed by fermion interactions.

By making H act on the left in equation (25) and by adding half of the two results, we find an expression of the H matrix element which is symmetrical with respect to the "in" and "out" states (i,j) and (m,n). It reads

$$\langle 0|B_m B_n H B_i^\dagger B_j^\dagger|0\rangle = (E_i+E_j)\delta_{m,i}\,\delta_{n,j} + \xi^{\rm dir}\left(\begin{smallmatrix} n & j \\ m & i \end{smallmatrix}\right) - \frac{1}{2}\left\{(E_m+E_n+E_i+E_j)\lambda\left(\begin{smallmatrix} n & j \\ m & i \end{smallmatrix}\right) + \xi^{\rm in}\left(\begin{smallmatrix} n & j \\ m & i \end{smallmatrix}\right) + \xi^{\rm out}\left(\begin{smallmatrix} n & j \\ m & i \end{smallmatrix}\right)\right\} + (m \leftrightarrow n), \tag{28}$$

where $\xi^{\rm out}$ is the exchange interaction scattering defined as

$$\xi^{\rm out}\left(\begin{smallmatrix} n & j \\ m & i \end{smallmatrix}\right) = \sum_{pq} \xi^{\rm dir}\left(\begin{smallmatrix} n & q \\ m & p \end{smallmatrix}\right) \lambda\left(\begin{smallmatrix} q & j \\ p & i \end{smallmatrix}\right), \tag{29}$$

and shown in Figure 5, its interactions being between the "out" cobosons (m,n).

The "in" and "out" exchange interaction scatterings are related to the Pauli scattering through [24,25]

$$\xi^{\rm in}\left(\begin{smallmatrix} n & j \\ m & i \end{smallmatrix}\right) - \xi^{\rm out}\left(\begin{smallmatrix} n & j \\ m & i \end{smallmatrix}\right) = (E_m+E_n-E_i-E_j)\lambda\left(\begin{smallmatrix} n & j \\ m & i \end{smallmatrix}\right). \tag{30}$$

They are thus equal for $E_m+E_n = E_i+E_j$, i.e., when the "in" and "out" cobosons exactly have the same energy.

2.5 The keys of the coboson many-body theory

The four commutators given in equations (13, 14) and (21, 22) are the keys to calculate any quantity involving interacting fermion pairs in the low density limit.

Indeed, by using equation (12), it is possible to write any N-pair state in terms of coboson operators. This makes appearing matrix elements like

$$\langle v|B_{m_N}\cdots B_{m_1}\, f(H)\, B^\dagger_{i_1}\cdots B^\dagger_{i_N}|v\rangle.$$

To calculate them, we first push $f(H)$ to the right by using the commutators $[f(H), B^\dagger]$ deduced from the commutator (21). The $f(H)$'s of physical interest are $1/(a-H)$ for correlation effects and response functions, and e^{-iHt} for time evolution or adiabatic establishment. The corresponding commutators can be found in references [29] and [30]. These commutators generate a lot of creation-potentials $V_i^\dagger$. They are eliminated through the commutator (22) to produce direct interaction scatterings $\xi^{\rm dir}\binom{n\ j}{m\ i}$. The remaining scalar products of N-coboson states are then calculated by using the commutator (13), the deviation-from-boson operators D_{mi} being eliminated through the commutator (14), to produce (2×2) Pauli scatterings $\lambda\binom{n\ j}{m\ i}$. It is actually possible to readily calculate these N-coboson scalar products in terms of n-body exchanges using the Shiva diagrams, as explained in reference [16].

Let us note that the first of these four commutators (13) has been written in the literature for a very long time by various authors. It is nothing but the mathematical way to write that the linear combinations of fermion pairs making $B_i^\dagger$ are not exact bosons. While the third of these commutators (21) is less trivial, it already appeared in the theory of the exciton optical Stark effect proposed by the author 20 years ago [31,32]. On the opposite, the two other commutators (14) and (22) are totally novel. They can be seen as a rather smart way to make appearing the scatterings resulting from interactions and fermion exchanges between two composite bosons.

2.6 The Hamiltonian expectation value

Using the above formalism based on the four commutators (13, 14, 21, 22), it is possible to calculate the expectation value of the system Hamiltonian in a state made of N atoms in the same state 0,

$$\langle H\rangle_N = \frac{\langle 0|B_0^N H B_0^{\dagger N}|0\rangle}{\langle 0|B_0^N B_0^{\dagger N}|0\rangle}, \tag{31}$$

in terms of the Pauli and Coulomb scatterings, λ and $\xi^{\rm dir}$. Let us stress that, while $B_0^{\dagger N}|0\rangle$ is definitely not the exact ground state of N atoms, it is close to it: following Keldysh and Koslov [33], this state leads to the expected N-atom ground state energy NE_0 at zero order in density — a result recovered below, in a trivial way.

Using equations (16, 28), it is straightforward to show that, for $N=2$, this expectation value is given by

$$\langle H\rangle_2 = 2\,\frac{2E_0\left[1-\lambda\binom{0\ 0}{0\ 0}\right]+\xi^{\rm dir}\binom{0\ 0}{0\ 0}-\xi^{\rm in}\binom{0\ 0}{0\ 0}}{2-2\lambda\binom{0\ 0}{0\ 0}}, \tag{32}$$

since $\xi^{\rm in}\binom{0\ 0}{0\ 0}=\xi^{\rm out}\binom{0\ 0}{0\ 0}$ according to equation (30). As $\xi^{\rm dir}\binom{0\ 0}{0\ 0}=0$ for fermions interacting through Coulomb potential, due to equation (24), this Hamiltonian expectation value for two H atoms or excitons thus reduces to

$$\langle H\rangle_2 = 2E_0 - \frac{\xi^{\rm in}\binom{0\ 0}{0\ 0}}{1-\lambda\binom{0\ 0}{0\ 0}}. \tag{33}$$

For 0 being their 3D ground state, $\xi^{\rm in}$, defined in equation (27), is equal to [34]

$$\xi^{\rm in}\binom{0\ 0}{0\ 0} = -\frac{26\pi}{3}\left(\frac{a_B}{L}\right)^3 R_0, \tag{34}$$

where $R_0 = e^2/2\varepsilon_r a_B$, the value of the diagonal Pauli scattering $\lambda\binom{0\ 0}{0\ 0}$ being given by equation (19). This shows that the energy change from the bare $2E_0$ value induced by the interactions of just 2 cobosons is vanishingly small when the sample volume goes to infinity, as physically reasonable.

If we now turn to the energy change induced by the interactions of N cobosons in the same state 0, we expect to have an N in front of these $(a_B/L)^3$ terms, i.e., to have an energy change depending on the atom density through the dimensionless parameter η defined in equation (1). A less trivial calculation [34] allowed us to show that

$$\langle H\rangle_N = N\Bigg[E_0 - \frac{(N-1)}{2}\frac{F_{N-2}}{F_N}\xi_2^{\rm in}\begin{pmatrix}0&0\\0&0\end{pmatrix} + (N-1)(N-2)\frac{F_{N-3}}{F_N}\xi_3^{\rm in}\begin{pmatrix}0&0\\0&0\\0&0\end{pmatrix}+\cdots\Bigg], \tag{35}$$

where $\xi_2^{\rm in}=\xi^{\rm in}$, while $\xi_3^{\rm in}$, which corresponds to the Shiva diagram of Figure 6, is equal to $(-\frac{2933}{20}\pi^2)\left(\frac{a_B}{L}\right)^6 R_0$; and so on... By using the recursion relation (18) for the F_N's, we end with the Hamiltonian expectation value for N atoms in the same ground state 0 with energy $E_0=-R_0$, given by

$$\langle H\rangle_N = NR_0\left[-1+\frac{13\pi}{3}\eta-\frac{73\pi^2}{2}\eta^2+O(\eta^3)\right]. \tag{36}$$

Note that the two first terms of this Hamiltonian expectation value have already been obtained through totally different procedures by Keldysh-Koslov [33] and by Nozières-Comte [35].

2.7 Overcompleteness of the coboson basis and closure relations

By using equations (11) and (12), it is possible to rewrite the product of two coboson operators in terms of all coboson states according to [24,36]

$$B_i^\dagger B_j^\dagger = -\sum_{mn} B_m^\dagger B_n^\dagger\,\lambda\binom{n\ j}{m\ i}. \tag{37}$$

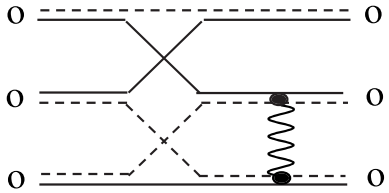

Fig. 6. Shiva diagram representing the exchange interaction scattering appearing in equation (35). It corresponds to fermion interactions between two cobosons in state 0 followed by a fermion exchange with a third coboson in state 0, the three scattered cobosons being in the same state 0 as the three initial cobosons.

This in particular leads to

$$B_0^{\dagger 2} = \frac{1}{1-\lambda \binom{0\ 0}{0\ 0}} \sum_{mn\neq 00} B_m^\dagger B_n^\dagger \, \lambda \binom{n\ 0}{m\ 0}, \qquad (38)$$

which proves that the coboson basis is overcomplete — and thus non orthogonal — for $N = 2$ cobosons already, since the state $B_0^{\dagger 2}|v\rangle$ can be written in terms of all the other 2-coboson states $B_m^\dagger B_n^\dagger |v\rangle$.

In spite of the fact that the N-coboson states form an overcomplete set for N fermion pairs, they do have a closure relation which turns out to be very simple. It reads [37]

$$I = \frac{1}{(N!)^2} \sum_{i_1,\ldots,i_N} B_{i_1}^\dagger \cdots B_{i_N}^\dagger |0\rangle\langle 0| B_{i_N} \cdots B_{i_1}. \qquad (39)$$

For comparison, the closure relation for elementary bosons, i.e., bosons such that $\left[\bar{B}_m, \bar{B}_i^\dagger\right] = \delta_{m,i}$, reads as

$$I = \frac{1}{N!} \sum_{i_1,\ldots,i_N} \bar{B}_{i_1}^\dagger \cdots \bar{B}_{i_N}^\dagger |0\rangle\langle 0| \bar{B}_{i_N} \cdots \bar{B}_{i_1}. \qquad (40)$$

It is of importance to stress that the factor $(1/N!)$ difference in the closure relations of composite and elementary bosons makes all the sum rules of these quantum particles irretrievably different whatever the bosonization procedure used to derive the effective scatterings between bosonized atoms is. This constitutes a very strong mathematical argument against the possible validity of *any* bosonization procedure which ends by using boson operators only. Let us also stress that the discrepancy exists for $N = 2$ already since the prefactor is then $1/4$ for composite bosons and $1/2$ for elementary bosons.

3 Survey of the Girardeau's boson-fermion procedure

We now turn to the procedure [7] developed by Girardeau in the 70's and which is still considered as an appropriate approach to interacting atoms. We are going to discuss this procedure in the light of the composite-boson many-body theory recalled in the preceding section. Equations prefaced with a "G" refer to the original 1975 paper, reference [7], in which this procedure is presented. We have chosen not to enter its improvements such as the one described in reference [11], to make clearer the basic ideas of this theory, in order to pin out the conceptual difficulties they raise.

Girardeau's paper starts with a system Hamiltonian, equation (G1), which is identical to the Hamiltonian of equations (5–8): it is just written in terms of electron field operators $\psi^\dagger(\mathbf{x})$ and proton field operators $\psi^\dagger(\mathbf{X})$, where $\mathbf{x}$ and $\mathbf{X}$ are the electron and proton spatial variables. These operators are related to the fermion operators $a_{\mathbf{k}_\alpha}^\dagger$ and $b_{\mathbf{k}_\beta}^\dagger$ introduced in equation (3) through the Fourier series

$$\psi^\dagger(\mathbf{x}) = \sum_{\mathbf{k}_\alpha} a_{\mathbf{k}_\alpha}^\dagger \, \langle \mathbf{k}_\alpha | \mathbf{x} \rangle, \qquad (41)$$

$$\psi^\dagger(\mathbf{X}) = \sum_{\mathbf{k}_\beta} b_{\mathbf{k}_\beta}^\dagger \, \langle \mathbf{k}_\beta | \mathbf{X} \rangle, \qquad (42)$$

with $\langle \mathbf{x}|\mathbf{k}\rangle = e^{i\mathbf{k}\cdot\mathbf{x}}/L^{D/2}$. Note that, while formally equivalent to the operators $a_{\mathbf{k}}^\dagger$ and $b_{\mathbf{k}}^\dagger$, the field operators $\psi^\dagger(\mathbf{x})$ and $\psi^\dagger(\mathbf{X})$ lead to equations which may appear as far less compact since the sums over the $\mathbf{k}$'s are then replaced by integrals over $(\mathbf{x}, \mathbf{X})$.

Although written in terms of free fermions only, the Hamiltonian H of course implicitly contains the fact that hydrogen atoms do have bound states through its one-pair eigenstates. In spite of it, Girardeau somehow forces these bound states into the problem via elementary-boson operators, by using a unitary transformation U which, in an exact way, transforms the one-composite-boson state $A_\alpha^\dagger|0\rangle$ for *bound* hydrogen atoms into the elementary-boson state $a_\alpha^\dagger|0\rangle$. In view of the definition of $A_\alpha^\dagger$ given in equation (G2), the correspondence between the two sets of notations reads

$$A_\alpha^\dagger \ \leftrightarrow \ B_i^\dagger \qquad (43)$$

$$a_\alpha^\dagger \ \leftrightarrow \ \bar{B}_i^\dagger. \qquad (44)$$

The only difference, of crucial importance, is for the index i which runs over all bound and unbound states in the composite-boson many-body theory, while the α's are by construction restricted to *bound* states only in the Girardeau's procedure. As an important consequence, the bound and unbound states $|i\rangle = B_i^\dagger|0\rangle$ form a complete set for one-fermion-pair states while this is not true for the $A_\alpha^\dagger|0\rangle$'s. This is why, in addition to the states $A_\alpha^\dagger|0\rangle$, Girardeau is forced to also keep the free-fermion-pair states $\psi^\dagger(\mathbf{X})\psi^\dagger(\mathbf{x})|0\rangle$ in order to possibly describe all the states made of one electron-proton pair reached by scatterings. In short, while in the composite-boson many-body theory, we have at hand two independent complete sets for one-fermion-pair states, namely, the $B_i^\dagger|0\rangle$'s and the $a_{\mathbf{k}_\alpha}^\dagger b_{\mathbf{k}_\beta}^\dagger|0\rangle$'s, the Girardeau's procedure makes use of all the $\psi^\dagger(\mathbf{X})\psi^\dagger(\mathbf{x})|0\rangle$ states — which form a complete basis by themselves — plus the $A_\alpha^\dagger|0\rangle$'s, these two sets of states then forming an overcomplete basis for one-fermion-pair states already, since $A_\alpha^\dagger$ reads in terms of $\psi^\dagger(\mathbf{X})\psi^\dagger(\mathbf{x})$

through equation (G3). It is clear that this overcompleteness is going to be tricky to handle properly.

The mapping from composite bosons to elementary bosons through the unitary transformation U, fully exact for one-pair bound states, becomes approximate when going beyond one-pair states. This is directly linked to the fact that, due to the overcompleteness of the composite-boson basis for any N larger than 1, it is not possible to separate products of bound-state atom operators from the unbound ones: indeed, as shown in equation (38), $B_0^{\dagger 2}$ reads in terms of all products of bound and unbound cobosons $B_m^\dagger B_n^\dagger$ with $(m,n) \neq (0,0)$. So that there is no clean way to separate a state only made of bound-atom operators $A_{\alpha_1}^\dagger ... A_{\alpha_N}^\dagger |0\rangle$ from the states also having unbound-atom operators: the separation between bound and unbound atom operators is one of the major conceptual difficulties encountered — and to our opinion not properly solved — by the procedure Girardeau has proposed.

The physical reason to separate bound atom operators from extended atom operators, is to have commutation rules as close to ideal bosons as possible. Since from equation (G5),

$$\left[A'_\alpha, A_\alpha^\dagger\right] = \delta_{\alpha',\alpha} + C_{\alpha'\alpha}, \tag{45}$$

this leads to require the effects of the operators $C_{\alpha'\alpha}$ to be small. In view of equation (13), the operator $C_{\alpha'\alpha}$ is just the operator $(-D_{mi})$ of the composite-boson many-body theory.

This operator $C_{\alpha'\alpha}$, explicitly given in equation (G6), reads in terms of the proton and electron field operators, $C_{\alpha'\alpha} = C_{\alpha'\alpha}^{(p)} + C_{\alpha'\alpha}^{(e)}$, as

$$C_{\alpha'\alpha}^{(p)} = -\int d\mathbf{X}\, d\mathbf{X}'\, K_{\alpha'\alpha}(\mathbf{X},\mathbf{X}')\, \psi^\dagger(\mathbf{X})\psi(\mathbf{X}'), \tag{46}$$

with $(\mathbf{X},\mathbf{X}')$ replaced by $(\mathbf{x},\mathbf{x}')$ in the case of $C_{\alpha'\alpha}^{(e)}$. The prefactors $K_{\alpha'\alpha}$, called "exchange kernels", depend on the atom wave functions through

$$K_{\alpha'\alpha}(\mathbf{X},\mathbf{X}') = \int d\mathbf{x}\, \phi_{\alpha'}^*(\mathbf{X}',\mathbf{x})\, \phi_\alpha(\mathbf{X},\mathbf{x}). \tag{47}$$

In bound states with a spatial extension a_B, the proton and the electron have to be at a distance of the order of a_B, for the atom wave function to differ from zero. Consequently, in order for the exchange kernel $K_{\alpha'\alpha}(\mathbf{X},\mathbf{X}')$, with α and α' being bound states, to differ from zero, we must have $\mathbf{x} \simeq \mathbf{X}' \simeq \mathbf{X}$ within a scale a_B. Instead of running over the whole sample volume L^3, the proton $\mathbf{X}'$ in equation (46) thus has to stay within a volume a_B^3 from the proton $\mathbf{X}$. This induces a factor a_B^3/L^3 in the operator $C_{\alpha'\alpha}$ which makes the matrix element of this operator in a N-atom state vanishing as $Na_B^3/L^3 = \eta$ with the atom density $n = N/L^3$. Consequently, the fact that the atoms differ from ideal bosons through the $C_{\alpha'\alpha}$ operators is going to induce corrections of the order of η in systems having N bound atoms.

3.1 Introduction of elementary-boson operators

In a second step, a new set of boson operators $a_\alpha^\dagger$ $(\equiv \bar{B}_i^\dagger)$ which act in a disjoint space, is formally introduced, and forced into the problem through the construction of the antihermitian operator $F = -F^\dagger$, defined as

$$F = \sum_{\alpha \text{ bound}} \left(a_\alpha^\dagger A_\alpha - A_\alpha^\dagger a_\alpha\right) \equiv \sum_{i \text{ bound}} \left(\bar{B}_i^\dagger B_i - B_i^\dagger \bar{B}_i\right), \tag{48}$$

the sum being restricted to bound states as the operators $a_\alpha^\dagger$ are only defined for these states. A unitary operator $U(\varepsilon)$ is then constructed out of F (see Eq. (G16)),

$$U(\varepsilon) = \exp(\varepsilon F), \tag{49}$$

where ε is a real constant yet arbitrary. As explicitly shown below, it is such that

$$U(\pi/2)\, A_\alpha^\dagger |0\rangle = a_\alpha^\dagger |0\rangle, \tag{50}$$

so that the unitary operator $U(\varepsilon)$, taken for the magic value $\varepsilon^* = \pi/2$, allows to in an *exact* way transform the one-composite-boson state $A_\alpha^\dagger|0\rangle$ into the one-elementary-boson state $a_\alpha^\dagger|0\rangle$. Let us note that if the sum over i in F were taken over bound *and* unbound states, this exact result would also be true for unbound states.

Unfortunately, this nicely simple result does not hold for more than one atom: the exact transformation of N-composite-atom states turns out to be very complicated for $N = 2$ already. Before going deeper into the problems raised by $N \geq 2$ atoms, let us remember that, according to Section 2,

$$\langle 0|B_0^N B_0^{\dagger N}|0\rangle = N! F_N = F_N \langle 0|\bar{B}_0^N \bar{B}_0^{\dagger N}|0\rangle, \tag{51}$$

with F_N exponentially small for large N, even if the atom density is very small. As, for $U(\varepsilon)$ being a unitary operator, this scalar product also reads

$$\langle 0|B_0^N B_0^{\dagger N}|0\rangle = \langle 0|B_0^N U^\dagger(\varepsilon) U(\varepsilon) B_0^{\dagger N}|0\rangle, \tag{52}$$

this means that the transformed N-composite-atom state $U(\varepsilon)B_0^{\dagger N}|0\rangle$ must be very different from the N-elementary-atom state $\bar{B}_0^{\dagger N}|0\rangle$ whatever the value of ε is, due to the exponentially small prefactor F_N in equation (51), the norm of $\bar{B}_0^{\dagger N}|v\rangle$ being $N!$ exactly.

By expanding $U(\varepsilon)$ in powers of F, it is possible to show that the transformed composite-atom creation operator $A_\alpha^\dagger \equiv B_i^\dagger$ reads as

$$U(\varepsilon) B_i^\dagger U^\dagger(\varepsilon) = B_i^\dagger + \frac{\varepsilon}{1!}[F, B_i^\dagger] + \frac{\varepsilon^2}{2!}[F,[F,B_i^\dagger]] + \frac{\varepsilon^3}{3!}[F,[F,[F,B_i^\dagger]]] + \cdots, \tag{53}$$

in agreement with equations (G22, G23). Using equations (13, 14), it is actually possible to write these commutators in terms of the deviation-from-boson operator

D_{mi} and the Pauli scatterings $\lambda\binom{n\ j}{m\ i}$. The commutators with one F only read

$$[F,\bar{B}^\dagger_{i\ \rm bound}] = -B^\dagger_i, \tag{54}$$

$$[F,B^\dagger_{i\ \rm bound}] = \bar{B}^\dagger_i - \sum_{m\ \rm bound} \bar{B}^\dagger_m D_{mi}, \tag{55}$$

$$[F,B^\dagger_{i\ \rm unbound}] = -\sum_{m\ \rm bound} \bar{B}^\dagger_m D_{mi}. \tag{56}$$

In order to get the commutators with two F, we first note that, due to equation (14),

$$[B_j,D_{mi}] = \sum_n \left[\lambda\binom{j\ n}{m\ i} + (i\leftrightarrow n)\right] B_n, \tag{57}$$

since $D^\dagger_{mi} = D_{im}$, while $\lambda\binom{n\ j}{m\ i}^* = \lambda\binom{j\ n}{i\ m}$. As $[D_{mi},\bar{B}^\dagger_j]=0$, for these operators act in different subspaces, we do have

$$[F,D_{mi}] = \sum_{j\ \rm bound}\sum_n \Big\{\left[\lambda\binom{j\ n}{m\ i} + (i\leftrightarrow n)\right]\bar{B}^\dagger_j B_n + \left[\lambda\binom{n\ j}{m\ i} + (m\leftrightarrow n)\right] B^\dagger_n \bar{B}_j\Big\}. \tag{58}$$

So that the commutator with two F, for bound composite atoms, reads

$$\left[F,\left[F,B^\dagger_{i\ \rm bound}\right]\right] = -B^\dagger_i + \sum_{m\ \rm bound} B^\dagger_m D_{mi} - \sum_{(m,j)\ \rm bound}\sum_n \bar{B}^\dagger_m\Big\{\bar{B}^\dagger_j B_n\left[\lambda\binom{j\ n}{m\ i} + (i\leftrightarrow n)\right] + B^\dagger_n\bar{B}_j\left[\lambda\binom{n\ j}{m\ i} + (m\leftrightarrow n)\right]\Big\}. \tag{59}$$

The calculation of commutators with more than two F are increasingly tedious and their expressions increasingly complicated. By noting that $D_{mi}|0\rangle = 0 = \bar{B}_j|0\rangle$, while $U^\dagger(\varepsilon)|0\rangle = |0\rangle$, it is however rather easy to iterate equation (59), in order to show that, for *one* bound atom only, we do have

$$U(\varepsilon)B^\dagger_{i\ \rm bound}|0\rangle = \left\{B^\dagger_i\left(1-\frac{\varepsilon^2}{2!}+\cdots\right) + \bar{B}^\dagger_i\left(\frac{\varepsilon}{1!}-\frac{\varepsilon^3}{3!}+\cdots\right)\right\}|0\rangle. \tag{60}$$

By identifying the above sums with $\cos\varepsilon$ and $\sin\varepsilon$, as somewhat reasonable, this shows that the one-composite-atom state $B^\dagger_i|v\rangle$ is exactly transformed into the one-ideal-atom state $\bar{B}^\dagger_{i\ \rm bound}|0\rangle$ by the unitary operator $U(\varepsilon)$ taken for the magic value $\varepsilon^* = \pi/2$.

For states having more than one atom, the operator $U(\varepsilon)$ does not allow us to stay within the bound-atom subspace, as can be seen from the sum over all n appearing in $[F,[F,B^\dagger]]$ already, as given in equation (59): this is just the signature of the fact that bound and unbound pairs of atoms are deeply linked by carrier exchanges — as seen from equation (37) —, so that the separation between these two types of operators is not mathematically possible. This also is fully consistent with the fact that, in addition to the bound-atom creation operator $A^\dagger_\alpha$, Girardeau has to keep the free-fermion-pair operators $\psi(\mathbf{x})^\dagger\psi(\mathbf{X})^\dagger$ in order to represent the unbound atom states possibly reached by scatterings.

Before going further, let us note that, according to equation (16) for 2 atoms, we must have

$$\langle 0|B^2_i U^\dagger(\varepsilon)U(\varepsilon)B^{\dagger 2}_i|0\rangle = \langle 0|B^2_i B^{\dagger 2}_i|0\rangle = 2 - 2\lambda\binom{i\ i}{i\ i}, \tag{61}$$

whatever ε is. This nicely compact result seems difficult to recover from the transformed two-atom state obtained by using the expansion (53), even if we choose the magic value $\varepsilon^* = \pi/2$, in order for $U(\varepsilon^*)B^\dagger_i|0\rangle$ to reduce to $\bar{B}^\dagger_i|0\rangle$. Indeed, this expansion then gives, according to equations (53–56), the transformed two-atom state as

$$U(\varepsilon^*)B^{\dagger 2}_{i\ \rm bound}|0\rangle = \Big\{U(\varepsilon^*)B^\dagger_i U^\dagger(\varepsilon^*)\Big\}\bar{B}^\dagger_i|0\rangle = \left[\left(1-\frac{\varepsilon^{*2}}{2!}\right)B^\dagger_i\bar{B}^\dagger_i + \frac{\varepsilon^*}{1!}\bar{B}^{\dagger 2}_i - \varepsilon^{*2}\sum_{m\ \rm bound,n}\lambda\binom{n\ i}{m\ i}\bar{B}^\dagger_m B^\dagger_n + O(\varepsilon^{*3})\right]|0\rangle, \tag{62}$$

for $D_{mi}\bar{B}^\dagger_i = 0 = B_n\bar{B}^\dagger_i|0\rangle$, as these operators act in different subspaces. By looking at the above result, we do not see how the term $2\lambda\binom{i\ i}{i\ i}$ in the norm of the transformed two-atom state can be recovered, even for the magic value ε^*. This difficulty encountered for $N=2$ atoms raises some major questions about the possibility to perform explicit calculations, even the simplest ones, within the Girardeau's approach.

In reference [7], the four operators A_α, a_α, $\psi(\mathbf{X})$ and $\psi(\mathbf{x})$ are transformed into $A_\alpha(\varepsilon) = U(\varepsilon)A_\alpha U^\dagger(\varepsilon)$ and similarly for $a_\alpha(\varepsilon)$, $\psi(\mathbf{X},\varepsilon)$ and $\psi(\mathbf{x},\varepsilon)$, not through equation (53) which seems hard to handle at high order, but by using an "equation-of-motion method", through the resolution of a set of coupled nonlinear differential equations for $a_\alpha(\varepsilon)$, $A_\alpha(\varepsilon)$, $\psi(\mathbf{X},\varepsilon)$ and $\psi(\mathbf{x},\epsilon)$. This set of equations is solved iteratively starting from the zero order solution obtained by completely neglecting all fermion exchanges between cobosons.

The transformed Hamiltonian $U(\varepsilon)HU^\dagger(\varepsilon)$ is then derived, starting from equation (2), in terms of the transformed free-fermion operators for $\varepsilon^* = \pi/2$, which is the magic value of ε giving a nicely simple result in the case of just one atom. This Hamiltonian, written in equation (G68), contains scatterings between free fermions, i.e., terms in $\psi^\dagger\psi^\dagger\psi\psi$, and scatterings between elementary bosons, i.e., terms in $a^\dagger a^\dagger aa$. It also contains very many terms in which one elementary ideal atom, scattered by free fermions or other elementary ideal atoms, dissociates into free fermions — or the reverse. The explicit calculation of the various terms appearing in the transformed

Hamiltonian gets very fast extremely heavy. Actually, in reference [7], the enumeration of these terms is limited to multiple collisions in which both the number of incoming and the number of outgoing particles is less or equal to three, in this way, de facto, truncating the transformed Hamiltonian.

3.2 Part of the transformed Hamiltonian UHU† acting in the ideal-atom subspace

In order to make some explicit comparison between the transformed Hamiltonian obtained by using Girardeau's Fock-Tani transformation described in reference [7] and the composite-boson many-body theory, we now concentrate on the part of the transformed Hamiltonian which only contains ideal-atom bosonic operators. Let us stress that Girardeau *never* advocates to truncate the transformed Hamiltonian to its ideal-atom part. However, in the spirit of bosonization, this truncation is quite tempting.

If we take as coboson α the exact one-pair eigenstate of the system Hamiltonian, the single-ideal-atom Hamiltonian given in equation (G69) reduces to

$$H_a = \sum_\alpha \epsilon_\alpha a_\alpha^\dagger a_\alpha \equiv \sum_{i \text{ bound}} E_i \bar{B}_i^\dagger \bar{B}_i. \tag{63}$$

If we turn to the interaction part between the two ideal atoms written in equation (G81),

$$H_{aa} = \frac{1}{2} \sum_{\alpha\beta\gamma\delta} a_\alpha^\dagger a_\beta^\dagger \langle \alpha\beta|H|\gamma\delta\rangle a_\gamma a_\delta, \tag{64}$$

it contains three contributions which have different physical origins. It is possible to write each of these contributions in terms of the 2×2 scatterings of the composite-boson theory.

The so-called "Coulomb contribution", given in equation (G82), is just the direct Coulomb scattering,

$$\langle \alpha\beta|H|\gamma\delta\rangle_{\text{Coul}} = \xi^{\text{dir}} \left(\begin{smallmatrix} \beta & \delta \\ \alpha & \gamma \end{smallmatrix}\right). \tag{65}$$

The second "Coulomb contribution", given in equation (G83), is nothing but the exchange Coulomb scattering symmetrized with respect to the "in" and "out" processes, namely,

$$\langle \alpha\beta|H|\gamma\delta\rangle_{\text{Coul-Ex}} = -\frac{1}{2}\left[\xi^{\text{in}} \left(\begin{smallmatrix} \beta & \delta \\ \alpha & \gamma \end{smallmatrix}\right) + \xi^{\text{out}} \left(\begin{smallmatrix} \beta & \delta \\ \alpha & \gamma \end{smallmatrix}\right)\right]. \tag{66}$$

If we now turn to the last contribution, given in equation (G84), it is easy to show that, for atom states being the exact eigenstates of the Hamiltonian, $[H(\mathbf{X},\mathbf{x}) - \epsilon_\gamma]\phi_\gamma(\mathbf{X},\mathbf{x}) = 0$, this contribution reduces to

$$\langle \alpha\beta|H|\gamma\delta\rangle_{\text{Intra-Ex}} = \frac{1}{6}(\epsilon_\alpha + \epsilon_\beta + \epsilon_\gamma + \epsilon_\delta)\, \lambda \left(\begin{smallmatrix} \beta & \delta \\ \alpha & \gamma \end{smallmatrix}\right). \tag{67}$$

By adding these three contributions, we end with

$$\langle \alpha\beta|H|\gamma\delta\rangle = \xi^{\text{dir}} \left(\begin{smallmatrix} \beta & \delta \\ \alpha & \gamma \end{smallmatrix}\right) - \frac{1}{2}\left[\xi^{\text{in}} \left(\begin{smallmatrix} \beta & \delta \\ \alpha & \gamma \end{smallmatrix}\right) + \xi^{\text{out}} \left(\begin{smallmatrix} \beta & \delta \\ \alpha & \gamma \end{smallmatrix}\right)\right] + \frac{1}{6}(\epsilon_\alpha + \epsilon_\beta + \epsilon_\gamma + \epsilon_\delta)\, \lambda \left(\begin{smallmatrix} \beta & \delta \\ \alpha & \gamma \end{smallmatrix}\right), \tag{68}$$

for the scattering of two ideal atoms from states (γ,δ) to states (α,β).

Since $\langle 0|a_\alpha a_\beta \psi^\dagger(\mathbf{X}) = 0$, for free fermions and ideal atoms act in different subspaces, the matrix element of the transformed Hamiltonian in the two-ideal-atom subspace thus reads

$$\begin{aligned}\langle 0|a_\alpha a_\beta U H U^\dagger a_\gamma^\dagger a_\delta^\dagger|0\rangle &= \langle 0|a_\alpha a_\beta (H_a + H_{aa}) a_\gamma^\dagger a_\delta^\dagger|0\rangle \\ &= (\epsilon_\alpha + \epsilon_\beta)\delta_{\alpha,\gamma}\,\delta_{\beta,\delta} + \xi^{\text{dir}} \left(\begin{smallmatrix} \beta & \delta \\ \alpha & \gamma \end{smallmatrix}\right) \\ &\quad - \frac{1}{2}\Bigg[- \frac{1}{3}(\epsilon_\alpha + \epsilon_\beta + \epsilon_\gamma + \epsilon_\delta)\lambda \left(\begin{smallmatrix} \beta & \delta \\ \alpha & \gamma \end{smallmatrix}\right) \\ &\quad + \xi^{\text{in}} \left(\begin{smallmatrix} \beta & \delta \\ \alpha & \gamma \end{smallmatrix}\right) + \xi^{\text{out}} \left(\begin{smallmatrix} \beta & \delta \\ \alpha & \gamma \end{smallmatrix}\right) \Bigg] \\ &\quad + (\alpha \leftrightarrow \beta).\end{aligned} \tag{69}$$

If we now compare this matrix element to the matrix element of the untransformed Hamiltonian in the two-composite-boson subspace, given in equation (28), we see that these two matrix elements are identical, except for the prefactor $(-1/3)$ in front of the $\epsilon\lambda$ terms. This prefactor is quite strange, not so much for the $1/3$, but mainly for its sign since, as ξ^{in} and ξ^{out}, this $\epsilon\lambda$ term contains *one* fermion exchange between two atoms, so that it should have the same sign as the Coulomb exchange terms. This $(-1/3)$ prefactor can be traced back to the scattering given in equation (67), in this way questioning the validity of the overall procedure to get $a_\alpha(\varepsilon)$ not through the exact equation (53) but through the "equation-of-motion method" which avoids the difficulty of calculating and mostly summing up these higher order commutators. However, since any physically relevant comparison between matrix elements has to be performed with normalized states, let us postpone this comparison to the next section.

4 Discussion

In this last section, we compare the composite-boson many-body theory to the Fock-Tani transformation procedure proposed by Girardeau in its original work [7], through three rather different points of view: aesthetics, simplicity and correctness.

4.1 Aesthetics

Instead of separating bound atom states from unbound ones and adding to the bound-atom states the full set of free-fermion states, as necessary to possibly represent dissociated states reached by scatterings, it seems to us far more natural to keep all the bound and unbound states of the atoms and to completely forget the free-fermion states, as done in the composite-boson many-body theory.

Since the free-fermion states $\psi^\dagger(\mathbf{X})\psi^\dagger(\mathbf{x})|0\rangle$ form a complete set for one-fermion-pair states in themselves, by adding the bound atom states $A_\alpha^\dagger|0\rangle$ to these free-fermion

states, Girardeau generates an overcomplete set of states for *one*-fermion-pair states already — which is an unnecessary source of difficulties. This has to be contrasted with the bound and unbound atom states $B_i^\dagger|0\rangle$, used in the composite-boson many-body theory: they form a complete set for one-fermion-pair states, so that no additional states are needed to describe any one-pair system.

We now turn to $N = 2$ fermion pairs. The states made of bound atoms plus free-fermion pairs as well as the states made of bound and unbound atoms, form an overcomplete set. However, by contrast to the bound-atom plus free-fermion states used by Girardeau, the overcomplete set of states made of bound and unbound atoms used in the coboson many-body theory is somewhat nicer as it has a closure relation, equation (39), very similar to the one for elementary bosons, except for the prefactor $1/4$ instead of $1/2$. This nicely simple closure relation is the neat signature of the fact that the ensemble of bound and unbound atoms is more natural to describe 2 atoms than the awkward ensemble made of bound atoms plus free-fermion pairs.

Actually, a clean separation between bound atom states and unbound atom states is not mathematically possible: Indeed a product of bound-atom operators can always be written in terms of unbound-atom operators, according to equation (37). In addition, this separation is not physically sharp since the difference between the highest bound states and the lowest unbound states is very weak. Consequently, it is not surprising to encounter difficulties when using such a separation based on so weak grounds. To our opinion, a more reasonable separation from a physical point of view would have been to isolate the atom ground state from the other bound and unbound states, although, again, this separation cannot be mathematically clean, due to equation (37).

4.2 Simplicity

The composite-boson many-body theory relies on the four nicely compact commutators written in equations (13, 14) and (21, 22). The last two ones are associated to interactions between cobosons without fermion exchange. The other two are associated to fermion exchanges without fermion interaction. All physical quantities dealing with interacting composite bosons ultimately read in terms of the Coulomb and Pauli scatterings generated by these four commutators, the calculation of any physical quantity involving N composite bosons reducing to a set of commutations, the ones for $N = 2$ atoms being totally trivial within this framework.

The procedure proposed by Girardeau in reference [7], based on a Fock-Tani transformation, is very attractive at first. Indeed, a smart unitary operator transforms a one-composite-atom bound state into a one-elementary-boson state, in an exact way, through an "intuitively straightforward" calculation (see Eq. (60)). Unfortunately, the transformation of N-composite-atom bound states for any N larger or equal to 2 turns out to be considerably more tedious. Being in fact done through an iterative process, it leads to a form which is not compact at all even for $N = 2$, i.e., when similar calculations done through the composite-boson theory are totally trivial. The transformed Hamiltonian this unitary operator generates, is also extremely complicated and impossible to write explicitly. It, in principle, contains an infinite number of terms which correspond to all possible scatterings between any number of bound atoms and free fermions. Its reduction to a finite number of terms — which is of practical necessity — is made through a reduction in the number of scattered quantum particles — atoms, free electrons or free protons — included in the theory. As a practical consequence, this limits the approach to problems involving a small amount of atoms. It is of importance to note that an order in the number of particles involved in the scatterings is *not at all* related to an order in the atom density — which is the key parameter of any many-body effects involving N atoms. Such a reduction thus appears as driven by the heaviness in the calculations we are ready to perform. On that respect, the Shiva-diagram representation of the composite-boson many-body theory is far nicer because such a density expansion is associated with a selection of diagrams with an increasing number of coboson lines — as can be for example seen from equation (35).

4.3 Correctness

Although not advocated by Girardeau, it is quite tempting to use his procedure as a bosonization procedure, i.e., to truncate the transformed Hamiltonian $UHU^\dagger$ to its part acting in the ideal-atom subspace, and to compare the result it generates for a physically relevant quantity to the one calculated with the untransformed Hamiltonian in the untransformed composite-atom subspace. This has the advantage to have a well defined transformed Hamiltonian instead of an infinite series of terms with no clear requirement for truncation.

Let us consider the Hamiltonian expectation value in a state made with N identical ground state atoms 0 and start with $N = 2$. Since for ideal (bosonic) atoms $\langle 0|a_0^N a_0^{\dagger N}|0\rangle = N!$, we get from equation (69) the expectation value of the transformed Hamiltonian for two ideal atoms as

$$\frac{\langle 0|a_0^2 UHU^\dagger a_0^{\dagger 2}|0\rangle}{\langle 0|a_0^2 a_0^{\dagger 2}|0\rangle} = 2E_0 + \frac{2}{3}E_0\lambda\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right) - \xi^{\rm in}\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), \quad (70)$$

since $\xi^{\rm dir}\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right) = 0$, according to equation (24), while $\xi^{\rm in}\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right) = \xi^{\rm out}\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)$, according to equation (30).

This result has to be contrasted with the exact expression of the Hamiltonian expectation value for two composite atoms, as trivially calculated from the composite-boson many-body theory, see equation (33),

$$2E_0 - \frac{\xi^{\rm in}\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)}{1-\lambda\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)}. \quad (71)$$

Since $\lambda\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right) = (33\pi/2)(a_B/L)^3$, as given in equation (19), is extremely small compared to 1 for macroscopic samples,

the denominator of equation (71) is essentially equal to 1. For $E_0 = -R_0$, the term $E_0\lambda \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)$ of equation (70) is of the order of $\xi^{\rm in}\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right) = -(26\pi/3)(a_B/L)^3 R_0$, as given in equation(34). So that, the results of equations (70) and (71) are definitely different, although the difference with the bare energy $2E_0$ is vanishing as $(a_B/L)^3$ in the large sample limit. This thus shows that one cannot replace composite atoms by the ideal bosonic atoms introduced by Girardeau, even for $N = 2$.

The discrepancy becomes physically relevant in the case of N atoms. Indeed, we have shown in equation (36) that the density expansion of the untransformed Hamiltonian expectation value reads as

$$\frac{\langle 0|B_0^N H B_0^{\dagger N}|0\rangle}{\langle 0|B_0^N B_0^{\dagger N}|0\rangle} \simeq N\left[E_0 - \frac{N}{2}\xi^{\rm in}\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right) + \cdots\right] \simeq NR_0\left[-1 + \frac{13\pi}{3}\eta + O(\eta^2)\right], \quad (72)$$

since $E_0 = -R_0$, in agreement with Keldysh-Koslov [33] and with Nozières-Comte [35].

Let us now consider a similar quantity in the ideal-atom subspace. Since the part of $UHU^\dagger$ acting on ideal atoms reduces to $H_a + H_{aa}$ given in equations (63, 64), its expectation value for N ideal atoms reads, using equation (69) and equations (19, 34),

$$\frac{\langle 0|a_0^N(H_a + H_{aa})a_0^{\dagger N}|0\rangle}{\langle 0|a_0^N a_0^{\dagger N}|0\rangle} \simeq N\left\{E_0 - \frac{N}{2}\left[\xi^{\rm in}\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right) - \frac{2}{3}E_0\lambda\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)\right]\right\} \simeq NR_0\left(-1 - \frac{7\pi}{6}\eta\right), \quad (73)$$

since $E_0 = -R_0$ for 3D ground state.

Besides the fact that it differs from the exact result given in equation (72), the above result has to be physically rejected for two major reasons:

(i) It would lead to a collapse since the energy of N atoms would decrease with increasing density.

(ii) The presence of a $E_0\lambda\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)$ term in the interacting part of the Hamiltonian expectation value, as given in equation (73), is physically unacceptable: it makes this interacting part depending on the choice made for the energy origin. In the case of excitons, the ground state energy E_0 is in fact equal to $(-R_0)$ plus the band gap; so that this $E_0\lambda\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)$ term would give a band gap contribution to the N-exciton energy coming from exciton-exciton interactions: it is physically clear that these interactions are only due to Coulomb interaction and carrier exchanges with no contribution from the carrier-ion interactions responsible for this band gap.

Actually, all these problems can be traced back to the replacement of composite bosons by elementary bosons — which is the aim of any bosonization procedure. In order to make it clear, let us come back to the case of just two atoms. When looking at equation (28), we see that a term in $E\lambda$ also exists in the matrix elements of the fermionic Hamiltonian H in the two-composite-atom subspace. However, in the Hamiltonian expectation value, this $E\lambda$ term is cancelled exactly by a similar Pauli term appearing in the scalar product of the two-composite-atom states, as seen from equation (33). When composite atoms are replaced by ideal (bosonic) atoms, the scalar products of the ideal-atom states do not contain any Pauli scattering, since the aim of bosonization is to forget all fermion exchanges once bosonization is performed. Consequently, the $E\lambda$ terms, which also exist in the matrix elements of the transformed Hamiltonian, cannot be cancelled when taking normalized two-ideal-atom states. This is the mathematical reason for the appearance of the spurious $E\lambda$ terms in the Hamiltonian expectation value, when restricted to ideal (bosonic) atoms.

It is of importance to stress that, if the unitary transformation were to be performed *exactly*, i.e., if this transformation were not used as a bosonization procedure by only keeping the ideal atom contribution, the two procedures should of course give exactly the same result, since the replacement of I by $U^{-1}U$ is totally armless. However, as

$$\frac{\langle 0|A_0^N H A_0^{\dagger N}|0\rangle}{\langle 0|A_0^N A_0^{\dagger N}|0\rangle} = \frac{[\langle 0|A_0^N U^\dagger(\varepsilon)]\,[U(\varepsilon)HU^\dagger(\varepsilon)]\,[U(\varepsilon)A_0^{\dagger N}|0\rangle]}{[\langle 0|A_0^N U^\dagger(\varepsilon)]\,[U(\varepsilon)A_0^{\dagger N}|0\rangle]}, \quad (74)$$

this exact calculation not only requires to use the exact transformed Hamiltonian $U(\varepsilon)HU^\dagger(\varepsilon)$, but also the exact transformed state $U(\varepsilon)A_0^{\dagger N}|0\rangle$. By approximating $U(\varepsilon^*)A_0^{\dagger n}|0\rangle$ by $a_0^{\dagger N}|0\rangle$, as done in equation (73), i.e., by only working in the ideal atom subspace [38] — which is the aim of any bosonization — we generate a major mistake, since the norm of this ideal-atom state is $N!$ instead of $N!F_N$. This mistake is the main reason for the unphysical result obtained in equation (73).

As the form of the transformed operator $A_\alpha^\dagger(\varepsilon)$ is not at all compact, it actually seems as extremely difficult to explicitly prove that the norm of $\left[A_\alpha^\dagger(\varepsilon)\right]^N|0\rangle$ is exactly equal to $N!F_N$, even in the case of just $N = 2$ atoms (see Eq. (62)). The exact expression of the transformed Hamiltonian also seems above any hope. Consequently, although it should in principle be possible to obtain correct results by performing the Fock-Tani unitary transformation exactly, calculations using this procedure are definitely far more complicated than the ones done by using the four nicely compact commutators of the composite-boson many-body theory given in equations (13, 14) and (21, 22), even in the "trivial" case of just 2 atoms.

Last but not least, the Shiva-diagram representation of this many-body theory, which is now available [16], allows an easy understanding of the fermion exchanges entering the various terms. This is quite valuable in view of the very tricky but quite interesting physics associated to these fermion exchanges, in particular when spin degrees of freedom are kept. By selecting diagrams with an

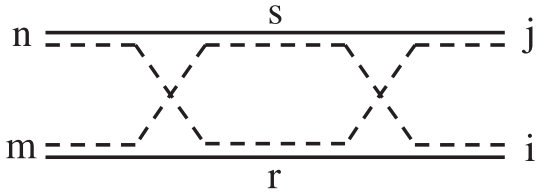

Fig. 7. Product of Pauli scatterings appearing in equation (76): it readily reduces to $\delta_{m,i}\,\delta_{n,j}$.

increasing number of coboson lines, it is possible to generate the density expansion of any physical effect at hand, while the selection of diagrams with an increasing number of interactions generates the detuning expansion of any optical nonlinearity in semiconductors.

4.4 Bosonization using a fully antisymmetrical coboson state

In this last section, let us come back to some consequences of the overcompleteness of the coboson basis which results from equation (37). Girardeau has suggested [5] to remove some unpleasant features of this overcompleteness by imposing a so-called "subsidiary condition". This corresponds to restrict to representations on the coboson basis which have stable prefactors under equation (37). As an example, among all the equivalent representations of the 2-composite-atom state $B_0^{\dagger 2}|v\rangle$ deduced from this equation (37), namely,

$$a\,B_0^{\dagger 2}|0\rangle - b\sum_{mn}\lambda\left(\begin{smallmatrix}n & 0\\ m & 0\end{smallmatrix}\right)B_m^\dagger B_n^\dagger|0\rangle, \qquad (75)$$

with $a+b=1$, we select the one with $a=b=1/2$: if we then replace the product of coboson operators by equation (37), the prefactors stay unchanged, as can be seen from

$$\begin{aligned}|\psi_{00}\rangle &= \frac{1}{2}B_0^{\dagger 2}|0\rangle - \frac{1}{2}\sum_{mn}\lambda\left(\begin{smallmatrix}n & 0\\ m & 0\end{smallmatrix}\right)B_m^\dagger B_n^\dagger|0\rangle\\ &= -\frac{1}{2}\sum_{pq}\lambda\left(\begin{smallmatrix}q & 0\\ p & 0\end{smallmatrix}\right)B_p^\dagger B_q^\dagger|0\rangle\\ &\quad+\frac{1}{2}\sum_{rs}B_r^\dagger B_s^\dagger|0\rangle\sum_{mn}\lambda\left(\begin{smallmatrix}s & n\\ r & m\end{smallmatrix}\right)\lambda\left(\begin{smallmatrix}n & 0\\ m & 0\end{smallmatrix}\right), \qquad (76)\end{aligned}$$

the second term reducing to $(1/2)B_0^{\dagger 2}|0\rangle$, since, as seen from Figure 7 or possibly checked through closure relations using equation (15), the sum of $\lambda\lambda$ appearing in equation (76) is equal to $\delta_{r,0}\,\delta_{s,0}$.

By imposing the stability condition with respect to equation (37), through $a=b=1/2$, we indeed remove the apparent indetermination which appears when writing a given two-pair state on the coboson basis. However, the introduction of this stable representation which may look very appealing at first, turns out to be a very bad idea. Indeed, following Barentzen and coworkers [9,39], we can be tempted to use this stable representation to make a one-to-one mapping between composite atoms and ideal atoms. This would lead to replace $B_0^{\dagger 2}|0\rangle$ not by $a_0^{\dagger 2}|0\rangle$ as done in equation (70), but by

$$|\overline{\psi}_{00}\rangle = \frac{1}{2}a_0^{\dagger 2}|0\rangle - \frac{1}{2}\sum_{mn}\lambda\left(\begin{smallmatrix}n & 0\\ m & 0\end{smallmatrix}\right)a_m^\dagger a_n^\dagger|0\rangle. \qquad (77)$$

If we then use the commutation rules for elementary bosons, it is easy to show that the normalization factor of the state $|\overline{\psi}_{00}\rangle$ now reads

$$\langle\overline{\psi}_{00}|\overline{\psi}_{00}\rangle = 1-\lambda\left(\begin{smallmatrix}0 & 0\\ 0 & 0\end{smallmatrix}\right). \qquad (78)$$

It now contains the Pauli scattering which was missing in $\langle 0|a_0^2 a_0^{\dagger 2}|0\rangle$, which is nice. It however differs by a factor of 2 from the bare normalization factor of the original state $\langle\psi_{00}|\psi_{00}\rangle$, given in equation (16), when Pauli scatterings are neglected — which is not too nice.

If we now calculate the diagonal matrix element of the most general effective Hamiltonian in the ideal-atom subspace, namely,

$$H^{\rm eff} = \sum_i \epsilon_i\,a_i^\dagger a_i + \frac{1}{2}\sum_{mnij}\overline{V}_{mnij}a_m^\dagger a_n^\dagger a_i a_j, \qquad (79)$$

we find for the 2-boson state $|\overline{\psi}_{00}\rangle$

$$\begin{aligned}\langle\overline{\psi}_{00}|H^{\rm eff}|\overline{\psi}_{00}\rangle &= \epsilon_0\left[1-2\lambda\left(\begin{smallmatrix}0 & 0\\ 0 & 0\end{smallmatrix}\right)\right]\\ &\quad+\sum\epsilon_i\,\lambda\left(\begin{smallmatrix}0 & j\\ 0 & i\end{smallmatrix}\right)\lambda\left(\begin{smallmatrix}j & 0\\ i & 0\end{smallmatrix}\right)+\frac{1}{2}\left[\overline{V}_{0000}-\sum\overline{V}_{00ij}\lambda\left(\begin{smallmatrix}j & 0\\ i & 0\end{smallmatrix}\right)\right.\\ &\quad\left.-\sum\lambda\left(\begin{smallmatrix}0 & n\\ 0 & m\end{smallmatrix}\right)\overline{V}_{mn00}+\sum\lambda\left(\begin{smallmatrix}0 & n\\ 0 & m\end{smallmatrix}\right)\overline{V}_{mnij}\lambda\left(\begin{smallmatrix}j & 0\\ i & 0\end{smallmatrix}\right)\right], \qquad (80)\end{aligned}$$

in which we have used $\overline{V}_{mnij}=\overline{V}_{nmij}=\overline{V}_{mnji}$, as always possible to enforce by symmetrizing equation (79).

We see, from equations (78, 80), that even the non-interacting part of the Hamiltonian expectation value calculated with $|\overline{\psi}_{00}\rangle$ instead of $a_0^{\dagger 2}|0\rangle$, is now incorrect: since, in the large sample limit, $\lambda\left(\begin{smallmatrix}0 & 0\\ 0 & 0\end{smallmatrix}\right)$ goes to zero, this non-interacting part now is ϵ_0 instead of $2\epsilon_0$.

The situation is somewhat worse if we use the bosonic Hamiltonian obtained through the Girardeau's procedure truncated to its ideal atom terms, as given in equation (69). Indeed, the $\epsilon\lambda$ term in the effective scattering modifies the leading term of this matrix element: As two exchanges reduce to an identity, we end with

$$\begin{aligned}\langle\overline{\psi}_{00}|UHU^\dagger|\overline{\psi}_{00}\rangle &= \frac{2}{3}\epsilon_0\left[1-2\lambda\left(\begin{smallmatrix}0 & 0\\ 0 & 0\end{smallmatrix}\right)\right]+\xi^{\rm dir}\left(\begin{smallmatrix}0 & 0\\ 0 & 0\end{smallmatrix}\right)\\ &\quad-2\xi^{\rm in}\left(\begin{smallmatrix}0 & 0\\ 0 & 0\end{smallmatrix}\right)+\frac{2}{3}\sum\epsilon_i\lambda\left(\begin{smallmatrix}0 & j\\ 0 & i\end{smallmatrix}\right)\lambda\left(\begin{smallmatrix}j & 0\\ i & 0\end{smallmatrix}\right)\\ &\quad+\sum\lambda\left(\begin{smallmatrix}0 & n\\ 0 & m\end{smallmatrix}\right)\xi^{\rm dir}\left(\begin{smallmatrix}n & j\\ m & i\end{smallmatrix}\right)\lambda\left(\begin{smallmatrix}j & 0\\ i & 0\end{smallmatrix}\right). \qquad (81)\end{aligned}$$

We thus conclude that the Hamiltonian expectation value obtained by using the form $|\overline{\psi}_{00}\rangle$ of the two-atom state, fully symmetrical with respect to exchange, is even more different from the exact value than the one obtained by simply using $a_0^{\dagger 2}|0\rangle$.

In addition, let us stress that the states (m,n) in the expansion (75) of $B_0^{\dagger 2}|0\rangle$ are bound as well as extended atomic states a priori. So that a direct mapping of $B_m^\dagger B_n^\dagger|0\rangle$ into $a_m^\dagger a_n^\dagger|0\rangle$ is not really possible for all (m,n) since these ideal bosonic states are defined for bound states only.

In the extended version [25] of our work on the exciton-exciton scattering rate [40], we have also considered the possibility to use Coulomb scatterings stable with respect to fermion exchanges. Due to equation (37), it reads

$$\begin{aligned}\sum_{mn}\xi^{\rm dir}\left(\begin{smallmatrix}n & j\\ m & i\end{smallmatrix}\right)B_m^\dagger B_n^\dagger &= a\sum_{mn}\xi^{\rm dir}\left(\begin{smallmatrix}n & j\\ m & i\end{smallmatrix}\right)B_m^\dagger B_n^\dagger\\ &\quad - b\sum_{pq}B_p^\dagger B_q^\dagger\sum_{mn}\lambda\left(\begin{smallmatrix}q & n\\ p & m\end{smallmatrix}\right)\xi^{\rm dir}\left(\begin{smallmatrix}n & j\\ m & i\end{smallmatrix}\right)\\ &= \sum_{mn}B_m^\dagger B_n^\dagger\left[a\,\xi^{\rm dir}\left(\begin{smallmatrix}n & j\\ m & i\end{smallmatrix}\right) - b\,\xi^{\rm in}\left(\begin{smallmatrix}n & j\\ m & i\end{smallmatrix}\right)\right], \quad (82)\end{aligned}$$

with $a+b=1$: we may think that the Coulomb scatterings between two cobosons are somewhat arbitrary. This is why we can also be tempted by taking scatterings which are stable with respect to carrier exchanges, these stable scatterings reading

$$\xi\left(\begin{smallmatrix}n & j\\ m & i\end{smallmatrix}\right) = \left[\xi^{\rm dir}\left(\begin{smallmatrix}n & j\\ m & i\end{smallmatrix}\right) - \xi^{\rm in}\left(\begin{smallmatrix}n & j\\ m & i\end{smallmatrix}\right)\right]/2. \quad (83)$$

In reference [16], we have explicitly shown that such "appealing" scatterings, when inserted in effective bosonic Hamiltonians, lead to results which are even farther from the correct values of the coboson transition rates and lifetime than the ones obtained from "unstable" Coulomb scatterings.

Before ending this paper, let us mention that, after the original work done by Girardeau in 1975, and used here to describe the basic ideas of this approach, developments have occured showing that this effective Hamiltonian misses some important physics, as for example, the orthogonality terms. Among them, we can cite the "resonating group method" described in reference [41] which does not rely on mapping procedures.

5 Conclusion

The aim of this work is to discuss the sophisticated procedure based on a unitary Fock-Tani transformation proposed [7] to describe interacting atoms by Girardeau in the mid 70's, in the light of the composite-boson many-body theory first developed for excitons [15] and recently extended to any type of composite bosons [42]. The Girardeau's procedure generates transformed states and transformed Hamiltonian so complicated that some truncation is a practical necessity. This can be done for example by restricting the approach to problems dealing with few body collisions. We here show that, when used as a bosonization procedure, i.e., when restricted to the ideal atom subspace in which atoms behave as elementary bosons — a restriction not advocated by Girardeau — this procedure leads to results which are definitely different from the exact ones, even in the extreme dilute limit of just two atoms. This work, once more, shows that the replacement of composite atoms by elementary quantum particles can be extremely dangerous, even qualitatively. In order to fully trust the obtained results, it seems to us far safer to stay in the original fermion space and to use the many-body theory for composite bosons which is now available. Its Shiva diagram representation [16] makes clear the subtle physics which results from the interplay between fermion interactions and fermion exchanges in such tricky composite-particle systems. These diagrams are also very valuable as they allow an easy selection and a readily calculation of pocesses leading to a density expansion or a detuning expansion of the physical effect at hand.

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